Energy and momentum of a spherically symmetric dilaton frame as regularized by teleparallel gravity*

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We calculate energy and momentum of a spherically symmetric dilaton frame using the gravitational energy-momentum 3-form within the tetrad formulation of general relativity (GR). The frame we use is characterized by an arbitrary function Υ with the help of which all the previously found solutions can be reproduced. We show how the effect of inertia *(which is mainly reproduced from* Υ*)* makes the total energy and momentum always different from the well known result when we use the Riemannian connection $\widetilde{\Gamma}_\alpha{}^\beta$. On the other hand, when use is made of the covariant formulation of teleparallel gravity, which implies to take into account the pure gauge connection, teleparallel gravity always yields the physically relevant result for the energy and momentum.

§1. Introduction

Our perspective and understanding of the universe have changed due to the new discoveries of the last decades. The discovery of the dark matter and the dark energy have opened new important questions about the nature of the matter in Cosmos. One of the accepted models to describe the nature of the dark energy is a scalar field model [1]. The dilaton is a scalar field occurring in the low energy limit of the theory where the Einstein action is supplemented by fields such as the axion, gauge fields and dilaton coupling in a nontrivial way to the other fields. Exact solutions for charged dilaton black holes in which the dilaton is coupled to the Maxwell field have been constructed by many authors. It is found that the presence of dilaton has important consequences on the causal structure and the thermodynamic properties of the black hole [2]∼[10]. Thus much interest has been focused on the study of the dilaton black holes.

* Keywords: gravitation, teleparallel gravity, energy-momentum, Weitzenböck connection, regularized teleparallel gravity

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Attempts at identifying an energy-momentum density for gravity has led to various energy momentum complexes which are pseudotensors [11]. Pseudotensors are not covariant objects i.e., they inherently depend on the reference frame, and thus cannot provide a true physical local gravitational energy-momentum density. Hence the pseudotensor approach has been largely abandoned (cf. [12]).

It is well known that teleparallel gravity theory allows a separation between gravitation and inertia [13]. Therefore, it turns out possible in this theory to write down a tensorial expression for the gravitational energy-momentum density [14]. Computation of the total energy of Schwarzschild and Kerr spacetimes using a regularized teleparallelism is given in [15]. Obukhov et al. [16] computed the energy and momentum transported by exact plane gravitational-wave solutions of Einstein equations using the teleparallel equivalent of general relativity (TEGR).

The aim of the present work is to calculate the energy and momentum of a general spherically symmetric dilaton frame with local Lorentz transformations containing an arbitrary function Υ which preserve spherical symmetry. Also we will show how inertia of energy and momentum are related to a pure gauge (Weitzenböck) connection ${\Gamma_\alpha}^{\beta}$*. In §2, we use the language of exterior forms to give an outline of the teleparallel approach. A brief review is given of the covariant formalism for the gravitational energy-momentum which is described by the pair $(\vartheta^\alpha, {\Gamma_\alpha}^\beta)$. In §3, we show by explicit calculations that due to an inconvenient choice of a reference frame, the traditional computation of the total energy and spatial momentum of the spherically symmetric dilaton solution are unphysical! Using the covariant formalism, we show that the Weitzenböck connection acts as a regularizing tool that separates the inertial contribution and always provides a physical meaningful result. The final section is devoted for main results and discussion.

Notation

We use the Latin indices $i, j, \cdots$ for local holonomic spacetime coordinates and the Greek indices α, β, $\cdots$ label (co)frame components. Particular frame components are denoted by hats, $\hat{0}$,$\hat{1}$, etc. As usual, the exterior product is denoted by $\wedge$, while the interior product of a vector ξ and a p-form Ψ is denoted by $\xi \rfloor \Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha$. Using local coordinates x^i, we have $\vartheta^\alpha = h^\alpha_i dx^i$ and $e_\alpha = h^i_\alpha \partial_i$ where h^α_i and h^i_α are the covariant and contravariant components of the tetrad field. We define the volume 4-form by $\eta \stackrel{\text{def.}}{=} \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}$. Furthermore, with the help of the interior product, we define

$$\eta_\alpha \stackrel{\text{def.}}{=} e_\alpha \rfloor \eta = \frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta}\, \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta,$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123} = 1$. Furthermore,

$$\eta_{\alpha\beta} \stackrel{\text{def.}}{=} e_\beta \rfloor \eta_\alpha = \frac{1}{2!} \epsilon_{\alpha\beta\gamma\delta}\, \vartheta^\gamma \wedge \vartheta^\delta, \qquad \eta_{\alpha\beta\gamma} \stackrel{\text{def.}}{=} e_\gamma \rfloor \eta_{\alpha\beta} = \frac{1}{1!} \epsilon_{\alpha\beta\gamma\delta}\, \vartheta^\delta,$$

which are bases for 3-, 2- and 1-forms respectively. Finally,

$$\eta_{\alpha\beta\mu\nu} \stackrel{\text{def.}}{=} e_\nu \rfloor \eta_{\alpha\beta\mu} = e_\nu \rfloor e_\mu \rfloor e_\beta \rfloor e_\alpha \rfloor \eta,$$

*We will use the same notation given in Ref. [15].

is the Levi-Civita tensor density. The η-forms satisfy the useful identities:

$$\vartheta^\beta \wedge \eta_\alpha = \delta^\beta_\alpha \eta, \qquad \vartheta^\beta \wedge \eta_{\mu\nu} = \delta^\beta_\nu \eta_\mu - \delta^\beta_\mu \eta_\nu, \qquad \vartheta^\beta \wedge \eta_{\alpha\mu\nu} = \delta^\beta_\alpha \eta_{\mu\nu} + \delta^\beta_\mu \eta_{\nu\alpha} + \delta^\beta_\nu \eta_{\alpha\mu},$$
$$\vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} = \delta^\beta_\nu \eta_{\alpha\gamma\mu} - \delta^\beta_\mu \eta_{\alpha\gamma\nu} + \delta^\beta_\gamma \eta_{\alpha\mu\nu} - \delta^\beta_\alpha \eta_{\gamma\mu\nu}. \tag{1}$$

The line element $ds^2 \stackrel{\text{def.}}{=} g_{\alpha\beta}\vartheta^\alpha \bigotimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$.

§2. Berif review of teleparallel gravity

Teleparallel geometry can be viewed as a gauge theory of translation [17]$\sim$[23]. The coframe ϑ^α plays the role of the gauge translational potential of the gravitational field. GR can be reformulated as the teleparallel theory. Geometrically, teleparallel gravity can be considered as a special case of the metric-affine gravity in which ϑ^α and the local Lorentz connection are subject to the distant parallelism constraint ${R_\alpha}^\beta = 0$ [24]$\sim$[33]. In this geometry the torsion 2-form

$$T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + {\Gamma_\beta}^\alpha \wedge \vartheta^\beta = \frac{1}{2}{T_{\mu\nu}}^\alpha \vartheta^\mu \wedge \vartheta^\nu = \frac{1}{2}{T_{ij}}^\alpha dx^i \wedge dx^j, \tag{2}$$

arises as the gravitational gauge field strength, ${\Gamma_\alpha}^\beta$ being the Weitzenbŏck 1-form connection, d is the exterior derivative and D is the exterior covariant derivative. The torsion T^α can be decomposed into three irreducible pieces [15], the tensor part, the trace and the axial trace given respectively by

$$\begin{aligned}
&{}^{(1)}T^\alpha \stackrel{\text{def.}}{=} T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha, \qquad \textit{with} \\
&{}^{(2)}T^\alpha \stackrel{\text{def.}}{=} \frac{1}{3}\vartheta^\alpha \wedge T, \quad \textit{where} \quad T = \left(e_\beta \rfloor T^\beta\right), \qquad e_\alpha \rfloor T = {T_{\mu\alpha}}^\mu, \quad \textit{vector of trace of torsion} \\
&{}^{(3)}T^\alpha \stackrel{\text{def.}}{=} \frac{1}{3}e^\alpha \rfloor P, \quad \textit{with} \quad P = \left(\vartheta^\beta \wedge T_\beta\right), \quad e_\alpha \rfloor P = T^{\mu\nu\lambda}\eta_{\mu\nu\lambda\alpha}, \quad \textit{axial of trace of torsion.}
\end{aligned} \tag{3}$$

The Lagrangian of the teleparallel equivalent of GR has the form

$$V = -\frac{1}{2\kappa} T^\alpha \wedge^* \left({}^{(1)}T_\alpha - 2^{(2)}T_\alpha - \frac{1}{2}{}^{(3)}T_\alpha\right), \tag{4}$$

$\kappa = 8\pi G/c^3$, G is the Newton gravitational constant, c is the speed of light and $*$ denotes the Hodge duality in the metric $g_{\alpha\beta}$ which is assumed to be flat Minkowski metric $g_{\alpha\beta} = o_{\alpha\beta} = diag(+1, -1, -1, -1)$, that is used to raise and lower local frame (Greek) indices.

The variation of the total action with respect to the coframe gives the field equations in the form

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \qquad \textit{where} \qquad \Sigma_\alpha \stackrel{\text{def.}}{=} \frac{\delta L_{mattter}}{\delta \vartheta^\alpha}, \tag{5}$$

is the canonical energy-momentum tensor 3-form of matter which is considered to be the source. In accordance with the general Lagrange-Noether scheme [20, 34], one derives from (4) the translational momentum 2-form and the canonical energy-momentum 3-form of the gravitational field:

$$H_\alpha \stackrel{\text{def.}}{=} -\frac{\partial V}{\partial T^\alpha} = \frac{1}{\kappa} * \left({}^{(1)}T_\alpha - 2{}^{(2)}T_\alpha - \frac{1}{2}{}^{(3)}T_\alpha\right), \tag{6}$$

$$E_\alpha \stackrel{\text{def.}}{=} \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \rfloor V + \left(e_\alpha \rfloor T^\beta\right) \wedge H_\beta. \tag{7}$$

Due to geometric identities [35], the Lagrangian (4) can be recast as

$$V = -\frac{1}{2} T^\alpha \wedge H_\alpha. \tag{8}$$

The presence of the connection field $\Gamma_\alpha{}^\beta$ plays an important regularizing role due to the following:

(i): The theory becomes explicitly covariant under local Lorentz transformations of the coframe, i.e., the Lagrangian (4) is invariant under the change of variables

$$\vartheta'^\alpha = \Lambda^\alpha{}_\beta \vartheta^\beta, \qquad \Gamma'^{\ \beta}_\alpha = \Lambda^\mu{}_\alpha \Gamma_\mu{}^\nu (\Lambda^{-1})^\beta{}_\nu - (\Lambda^{-1})^\beta{}_\gamma d\Lambda^\gamma{}_\alpha. \tag{9}$$

Due to the non-covariant transformation law of $\Gamma_\alpha{}^\beta$, see Eq. (9), if a connection vanishes in a given frame, it will not vanish in any other frame related to the first by a local Lorentz transformation.

(ii): $\Gamma_\alpha{}^\beta$ plays an important role in the teleparallel framework. This role represents the inertial effects which arise from the choice of the reference system [14]. The contributions of this inertial object in many cases lead to unphysical results for the total energy of the system. Therefore, the role of the teleparallel connection, is to separate the inertial contribution from the truly gravitational one. Since the teleparallel curvature is zero, the connection is a pure gauge, that is

$$\Gamma_\alpha{}^\beta = (\Lambda^{-1})^\beta{}_\gamma d\Lambda^\gamma{}_\alpha. \tag{10}$$

The Weitzenb*ö*ck connection always has the form (10). The translational momentum has the form [15]

$$\widetilde{H}_\alpha = \frac{1}{2\kappa} \widetilde{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}, \qquad \Gamma_\alpha{}^\beta \stackrel{\text{def.}}{=} \widetilde{\Gamma}_\alpha{}^\beta - K_\alpha{}^\beta, \tag{11}$$

with $\widetilde{\Gamma}_\alpha{}^\beta$ is the purely Riemannian connection and $K^{\mu\nu}$ is the contortion 1-form which is related to the torsion through

$$T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta. \tag{12}$$

§3. Total energy of spherically symmetric dilaton spacetime

In this section, we are going to show how the Weitzenböck connection ${\Gamma_\alpha}^\beta$ acts as an inertial object and contributes to the physical quantities like energy, momentum etc., when it is trivial. On the other hand, when this connection is non trivial, we show how it separates the inertia from the other physical quantities. We will show this by studying a spherically symmetric spacetime which contains an arbitrary function Υ which preserve spherical symmetry and reproduce all the previous solutions. *This study is carried out for the spherically symmetric cases only.*

Using the spherical local coordinates (t, r, θ, ϕ), the spherically symmetric dilaton is described by the coframe components:

$$\overset{S}{\vartheta}{}^{\alpha} = {\Lambda_1}^\alpha{}_\gamma {\Lambda_2}^\gamma{}_\delta \vartheta^\delta, \tag{13}$$

where the coframe ϑ^δ has the form

$$\vartheta^{\hat{0}} = \alpha^{-1} c dt, \quad \vartheta^{\hat{1}} = \alpha dr, \quad \vartheta^{\hat{2}} = r\ \beta\ d\theta, \quad \vartheta^{\hat{3}} = r\ \beta\ \sin\theta\ d\phi, \qquad \textit{where}$$
$$\alpha = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}, \qquad \beta = \sqrt{1 - \frac{q^2 e^{-2\phi_0}}{mr}}, \tag{14}$$

where m, q and ϕ_0 are the mass, the charge and the asymptotic value of the dilaton, respectively. The matrices ${\Lambda_1}^\alpha{}_\gamma$ and ${\Lambda_2}^\gamma{}_\delta$ are the local Lorentz transformations that are defined respectively as

$${\Lambda_1}^\alpha{}_\gamma \stackrel{\text{def.}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix}, \tag{15}$$

$${\Lambda_2}^\gamma{}_\delta \stackrel{\text{def.}}{=} \begin{pmatrix} \beta_1 & \beta_2 \sin\theta\cos\phi & \beta_2\sin\theta\sin\phi & \beta_2\cos\theta \\ -\beta_2\sin\theta\cos\phi & 1+\beta_3\sin^2\theta\cos^2\phi & \beta_3\sin^2\theta\sin\phi\cos\phi & \beta_3\sin\theta\cos\theta\cos\phi \\ -\beta_2\sin\theta\sin\phi & \beta_3\sin^2\theta\sin\phi\cos\phi & 1+\beta_3\sin^2\theta\sin^2\phi & \beta_3\sin\theta\cos\theta\sin\phi \\ -\beta_2\cos\theta & \beta_3\sin\theta\cos\theta\cos\phi & \beta_3\sin\theta\cos\theta\sin\phi & 1+\beta_3\cos^2\theta \end{pmatrix},$$
$$\textit{with} \qquad \beta_1 = 1 + \sqrt{e^{2\Upsilon}+1} - \sqrt{2},$$
$$\beta_2 = \sqrt{3 + 2\sqrt{e^{2\Upsilon}+1}\left(1-\sqrt{2}\right) + e^{2\Upsilon} - 2\sqrt{2}}, \qquad \beta_3 = \beta_1 - 1, \tag{16}$$

where $\Upsilon = \Upsilon(r)$ is an arbitrary function. It can be shown that $\Upsilon(r)$ can reproduce the previous arbitrary function $H(r)$ studied in ([36], Eq. (16)) through the relation*

$$\Upsilon(r) = \ln\sqrt{2\sqrt{H^2(r)+1}(\sqrt{2}-1) + H^2(r) + 3 - 2\sqrt{2}}\ . \tag{17}$$

*When $H(r) = 0$, Eq. (17) gives $\Upsilon = 0$ and Eq. (16) gives the identity matrix which will be identical with Eq. (3·7) in [36], Eq. (16) in Ref. [37] and Eq. (13) in [38] when we used the proper Lorentz transformation.

The metric tensor $g_{ij} \stackrel{\text{def.}}{=} o_{\mu\nu}h^\mu{}_i h^\nu{}_j$ associated with the tetrad field (13) has the form

$$ds^2 = \frac{1}{\alpha^2}dt^2 - \alpha^2 dr^2 - r^2\beta^2 d\Omega^2, \quad with \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

which is dilaton spacetime derived in [3].

From Eq. (17) all the previous spherically symmetric solution can be obtained [38]. If we take tetrad (13), as well as the trivial Weitzenböck connection $\Gamma^\alpha{}_\beta = 0$ and substitute into (11), we finally get

$$\begin{aligned}\widetilde{H}_{\hat{0}} = & -\frac{\sin\theta\beta r}{8\pi}\Bigg\{\left(\cos\phi\sin\theta\cos\theta\sin\phi + \cos^2\phi\sin^2\theta + \cos\theta\sin\phi + 2\frac{\beta' r+\beta}{\alpha} - 1\right)\beta_3 \\ & -2\left(1 - \frac{\beta' r+\beta}{\alpha}\right)\Bigg\}d\theta\wedge d\phi. \end{aligned} \tag{18}$$

If we compute the total energy at a fixed time in the 3-space with a spatial 2-dimensional boundary surface $\partial S = \{r = R, \theta, \phi\}$, we obtain

$$\widetilde{E} = \int_{\partial S} \widetilde{H}_{\hat{0}} = \frac{2R\beta}{3}\left(\beta_1\left[1 - \frac{3[\beta'+\beta]}{\alpha}\right]\right). \tag{19}$$

Using Eq. (19) we discuss the following cases:

(i) When $\Upsilon(R) = 0$, then Eq. (16) gives $\beta_1 = 1$, $\beta_2 = 0$ and $\beta_3 = 0$. In this case the energy takes the form up to $O\left(\frac{1}{R}\right)$

$$\widetilde{E} \cong m - \frac{q^2e^{-2\phi_0} - m^2}{2R} - \frac{q^4e^{-4\phi_0}}{8m^2R} + O\left(\frac{1}{R^2}\right), \tag{20}$$

which is consistent with the previous result ([39], Eq (45)). In this case the local Lorentz transformation given by Eq. (16) will be identical with the Kronecker delta, i.e., $\delta^\alpha_\beta = diag(+1, +1, +1, +1)$.

(ii) When $\Upsilon(R) \cong \dfrac{c_1}{\sqrt{R}}$ we get

$$\widetilde{E} = -\frac{c_1\sqrt{2R}}{3} + m - \frac{{c_1}^2}{2\sqrt{2}} + \frac{c_1q^2e^{-2\phi_0} + 3c_1}{3m\sqrt{2R}} - \frac{q^2e^{-2\phi_0} - m^2}{2R} - \frac{q^4e^{-4\phi_0}}{8m^2R} + O\left(\frac{1}{R^{3/2}}\right), \tag{21}$$

which is a divergent one. It is clear from Eq. (21) how the inertia c_1 contributes the physical quantities.

(iii) When $\Upsilon(R) \cong \dfrac{c_1}{R}$, then the energy takes the form

$$\widetilde{E} = m - \frac{c_1\sqrt{2}}{3} + \frac{2c_1q^2e^{-2\phi_0} + 3{c_1}^2 - 3m\sqrt{2}q^2e^{-2\phi_0} + 3m^3\sqrt{2}}{6m\sqrt{2R}} + O\left(\frac{1}{R}\right), \tag{22}$$

which is not divergent but not consistence with the previous results ([39], Eq. (72)). If we continue in this manner, i.e., $\Upsilon(R) \cong \frac{c_1}{R^{3/2}}$ or $\Upsilon(R) \cong \frac{c_1}{R^2}$, we can show that the inertia will continue in its contribution to the physical quantities up to order $O\left(\frac{1}{R^2}\right)$.

(iv) When $\Upsilon(R) \cong \frac{c_1}{R^{5/2}}$ the form of energy will be the same as given by Eq. (20).

To overcome the above problems (divergent or contribution of inertia to physical quantities) we are going to use the regularization framework which is based on the covariance property, i.e., we will take into account the Weitzenb$\ddot{o}$ck connection ${\Gamma^\alpha}_\beta$ given by Eq. (10) in which ${\Lambda^\alpha}_\beta = {\Lambda_1}^\alpha{}_\gamma ({\Lambda_2}^{-1})^\gamma{}_\delta ({\Lambda_1}^{-1})^\delta{}_\beta$ [15].

Using the regularization framework and calculating the necessary components, we finally get the superpotential

$$H_{\hat{0}} = \frac{\beta r \sin\theta}{4\pi}\left[\left\{\sqrt{e^{2\Upsilon}+1}(\sqrt{2}-1) - e^{2\Upsilon} - 1\right\}(\beta' r + \beta - \alpha)\right] d\theta \wedge d\phi. \tag{23}$$

The total energy of (23) thus has the form

$$E = \int_{\partial S} H_{\hat{0}} = R\beta\left[\left\{\sqrt{e^{2\Upsilon}+1}(\sqrt{2}-1) - e^{2\Upsilon} - 1\right\}(\beta' r + \beta - \alpha)\right]. \tag{24}$$

From Eq. (24) we discuss the following:

If $\Upsilon(R) = 0$ or $\Upsilon(R) = \frac{c_1}{\sqrt{R}}$ or $\Upsilon(R) = \frac{c_1}{R}$ etc., the value of the energy will have the form of Eq. (20) which is consistence with the previous results ([39] Eq. (45))†.

The non vanishing components needed to calculate the spatial momentum $\widetilde{H}_{\hat{\alpha}} = H_{\hat{\alpha}}$, $\hat{\alpha} = 1, 2, 3$ have the form

$$\widetilde{H}_{\hat{1}} = H_{\hat{1}} = \frac{\beta\beta_2 r(\beta' r + \beta - \alpha)\sin\theta\left[\sin\phi\cos\phi\{1 - \sin\theta\cos\theta\} - \cos^2\phi\sin^2\phi\right]}{4\alpha\pi} d\theta \wedge d\phi,$$
$$\widetilde{H}_{\hat{2}} = H_{\hat{2}} = \frac{\beta\beta_2 r(\beta' r + \beta - \alpha)\sin\theta\left[\cos\theta\cos\phi(\sin\theta\cos\phi - 1) - \cos\phi\sin\phi\sin^2\theta - \cos\theta\sin\theta\right]}{4\alpha\pi} d\theta \wedge d\phi,$$
$$\widetilde{H}_{\hat{3}} = H_{\hat{3}} = \frac{\beta\beta_2 r(\beta' r + \beta - \alpha)\sin^2\theta(\cos\theta\cos\phi - \sin\theta\sin\phi)}{4\alpha\pi} d\theta \wedge d\phi. \tag{25}$$

Using Eqs. (25) in Eq. (11), we finally get the spatial momentum in the form

$$P_{\hat{\alpha}} = \int_{\partial S} H_{\hat{\alpha}} = 0, \qquad \hat{\alpha} = 1, 2, 3. \tag{26}$$

†Quadratic terms like c_1m are neglected in this approximation.

§4. Main results and Discussion

The main results of this paper are the following:

• Of new local Lorentz transformation with an arbitrary function $\Upsilon(r)$ which maintain spherical symmetry is given by Eq. (16). The relation between this transformation and the transformation studied in [38] is given through Eq. (17).

• The frame we have studied creates the same spacetime as the one derived in [3]. This spacetime is characterize by the gravitational mass m, the charge q and the asymptotic value of the dilaton ϕ_0.

• We have calculated the energy of the frame (13) by using two procedures:

(i) In the first procedure, we have taken the Riemannian connection only and we have shown that the energy may be divergent or not be of the well know form. We have explained how the form of energy depends on the asymptotic value of the arbitrary function $\Upsilon(R)$. When the arbitrary function $\Upsilon(R)$ is asymptotically smaller than $\frac{1}{R^{5/2}}$, then the form of energy may be divergent or not in agreement with the previous result ([39], Eq. (45)). On the other hand, if $\Upsilon(R)$ is asymptotically greater or equal to $\frac{1}{R^{5/2}}$, the form of energy will be in agreement with the previous result [38].

(ii) When use is made of the covariant formulation of teleparallel gravity, which implies to take into account the pure gauge (or Weitzenböck) connection given in Eq. (10). Teleparallel gravity always yields the physically relevant result for the energy and momentum.

• The second argument in the previous item is known as the regularization of the teleparallelism in which the covariant teleparallel approach always yields the physically correct result.

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A comparison between the results of the arbitrary function $\Upsilon(r)$ which reproduces the dilaton spacetime and its energy using different translational momenta are given in the following table.

Table 1. Comparison between the results of the arbitrary function $\Upsilon(r)$ which keeps spherical symmetry and the other constants which create dilaton spacetime and its energy using different translational momenta

Spacetime	Arbitrary function $\Upsilon(r)$	The constant c_1	Translational Momentum	Energy & References
Schwarzschild	$\Upsilon(r)=0$, $q=0$ in Eq. (13)	$c_1=0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E=m$, Eq. (20), Ref. [36]
	$\Upsilon(r)\neq 0$, $q=0$ in Eq. (13)	$c_1=m$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E\neq m$, Eq. (21), Ref. [36]
	$\Upsilon(r)\neq 0$, $q=0$ in Eq. (13)	$c_1\neq 0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E\neq m$, Eq. (21)
	$\Upsilon(r)\neq 0$, $q=0$ in Eq. (13)	$c_1\neq 0$	Equation (11) with non trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}\neq 0$	$E=m$, Eq. (24)
Reissner-Nordström	$\Upsilon(r)=0$	$c_1=0$ $\phi_0=0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E=m-\frac{q^2}{2r}$, Eq. (20), Ref. [36]
	$\Upsilon(r)\neq 0$	$c_1=m$ $\phi_0=0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E\neq m-\frac{q^2}{2r}$, Ref. [36]
	$\Upsilon(r)\neq 0$	$c_1\neq 0$ $\phi_0=0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E\neq m-\frac{q^2}{2r}$, Eq. (20)
	$\Upsilon(r)\neq 0$	$c_1\neq 0$ $\phi_0=0$	Equation (11) with non trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}\neq 0$	$E=m-\frac{q^2}{2r}$, Eq. (24)
Dilaton spacetime	$\Upsilon(r)=0$	$c_1=0$ $\phi_0\neq 0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E=m-\frac{q^2e^{-2\phi_0}-m^2}{2R}$, Ref. [36]
	$\Upsilon(r)\neq 0$	$c_1=m$ $\phi_0\neq 0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E\neq m-\frac{q^2e^{-2\phi_0}-m^2}{2R}$, Ref. [36]
	$\Upsilon(r)\neq 0$	$c_1\neq 0$ $\phi_0\neq 0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E\neq m-\frac{q^2e^{-2\phi_0}-m^2}{2R}$, Eq. (21)
	$\Upsilon(r)\neq 0$	$c_1\neq 0$ $\phi_0\neq 0$	Equation (11) with non trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}\neq 0$	$E=m-\frac{q^2e^{-2\phi_0}-m^2}{2R}$, Eq. (24)

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